NBI-HE-97-31
July 1997

Explicit Decompositions of Weyl Reflections in Affine Lie Algebras

Jørgen Rasmussen[1]

The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

In this paper explicit decompositions are provided of the Weyl reflections in affine Lie algebras, in terms of fundamental Weyl reflections.

PACS: 02.20.Sv; 11.25.Hf
Keywords: Affine Lie algebra; Lie algebra; Weyl group

[1]e-mail address: jrasmussen@nbi.dk

1 Introduction

An understanding of the Weyl group of an affine Lie algebra resides in the basis of affine Lie algebra theory [1]. Just as in the case of the usual finite dimensional Lie algebras, the (affine) Weyl group is fundamental in discussions on e.g. characters. In applications of the Weyl group it is sometimes of importance to be able to decompose the Weyl reflections into products of fundamental reflections which are reflections with respect to simple roots. This is the case when considering singular vectors along the lines of Malikov, Feigin and Fuks [2].

The main result in this paper is the presentation of explicit decompositions into fundamental reflections of all Weyl reflections in all affine Lie algebras based on simple finite dimensional Lie algebras of the types $A_r,\ B_r,\ C_r,\ D_r,\ E_6,\ E_7,\ E_8,\ F_4$ and G_2. The decompositions we present rely on a new universal lemma and on well known explicit, algebra dependent realizations of the root systems in the associated finite dimensional Lie algebras (see e.g. [3]). The lemma reduces the problem to an associated problem of decomposing classical Weyl reflections. By classical Weyl reflections we mean reflections in a standard finite dimensional Lie algebra. Explicit decompositions of the classical Weyl reflections are also worked out. To the best of our knowledge, explicit decompositions of affine Weyl reflections are only known in a few examples, see e.g. [4, 5].

Our motivation for considering explicit decompositions of affine Weyl reflections is the wish to use them in a study of fusion rules in conformal field theories based on affine current algebras [6]. In that respect explicit decompositions are necessary in order to generalize a work by Awata and Yamada [7] on fusion rules for affine $SL(2)$ current algebra. Recently, their approach has been used to determine the fusion rules for admissible representations [8, 9] of affine $OSp(1|2)$ current algebra [10]. Moreover, one needs differential operator realizations of the underlying finite dimensional Lie algebras. In the work [11, 12] by Petersen, Yu and the present author, such realizations have been worked out for all simple finite dimensional Lie algebras. The decompositions presented in this paper then allow generalizing the work [7].

The remaining part of the paper is organized as follows. Section 2 serves to fix notation. Some basic Lie algebra and affine Lie algebra properties are reviewed. In Section 3 explicit decompositions of affine Weyl reflections are worked out first in terms of simple reflections and classical Weyl reflections of the associated finite dimensional Lie algebra. The classical Weyl reflections are then decomposed into fundamental reflections. Section 4 contains concluding remarks, while explicit, algebra dependent realizations of root systems in finite dimensional Lie algebras are reviewed in Appendix A.

2 Notation

2.1 Lie Algebras

Let $\mathbf{g}$ be a simple Lie algebra of rank $\mathbf{g} = r$. $\mathbf{h}$ is a Cartan subalgebra of $\mathbf{g}$. The set of (positive) roots is denoted $(\Delta_+)\ \Delta$, and we write $\alpha > 0$ if $\alpha \in \Delta_+$. The simple roots are $\{\alpha_i\}_{i=1,\ldots,r}$. θ is the highest root, while $\alpha^\vee = 2\alpha/\alpha^2$ is the root dual to α. Using the

triangular decomposition

$$\mathbf{g} = \mathbf{g}_- \oplus \mathbf{h} \oplus \mathbf{g}_+ \tag{1}$$

the raising and lowering generators are denoted $e_\alpha \in \mathbf{g}_+$ and $f_\alpha \in \mathbf{g}_-$ respectively with $\alpha \in \Delta_+$, and $h_i \in \mathbf{h}$ are the Cartan generators. We let j_a denote an arbitrary Lie algebra generator. For simple roots one sometimes writes $e_i = e_{\alpha_i}, f_i = f_{\alpha_i}$. The $3r$ generators e_i, h_i, f_i are the Chevalley generators. Their commutator relations are

$$\begin{aligned} [h_i, h_j] &= 0 \qquad & [e_i, f_j] &= \delta_{ij} h_j \\ [h_i, e_j] &= A_{ij} e_j \qquad & [h_i, f_j] &= -A_{ij} f_j \end{aligned} \tag{2}$$

where A_{ij} is the Cartan matrix. In the Cartan-Weyl basis we have

$$[h_i, e_\alpha] = (\alpha_i^\vee, \alpha) e_\alpha \qquad , \qquad [h_i, f_\alpha] = -(\alpha_i^\vee, \alpha) f_\alpha \tag{3}$$

and

$$[e_\alpha, f_\alpha] = h_\alpha = G^{ij}(\alpha_i^\vee, \alpha^\vee) h_j \tag{4}$$

where the metric G_{ij} is related to the Cartan matrix as $A_{ij} = \alpha_i^\vee \cdot \alpha_j = (\alpha_i^\vee, \alpha_j) = G_{ij}\alpha_j^2/2$, while the non-vanishing elements of the Cartan-Killing form are

$$\kappa_{\alpha,-\beta} = \kappa(e_\alpha f_\beta) = \frac{2}{\alpha^2}\delta_{\alpha,\beta} \qquad , \qquad \kappa_{ij} = \kappa(h_i h_j) = G_{ij} \tag{5}$$

The Weyl reflections σ_α acting on $\lambda \in \mathbf{h}^*$ (or equivalently on $\beta \in \Delta$) are defined by

$$\sigma_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha \tag{6}$$

and are generators of the Weyl group. It is easily seen that

$$\sigma_\alpha^{-1} = \sigma_\alpha \qquad , \qquad \sigma_{-\alpha} = \sigma_\alpha \qquad , \qquad \sigma_{\sigma_\alpha(\beta)} = \sigma_\alpha \circ \sigma_\beta \circ \sigma_\alpha \tag{7}$$

2.2 Affine Lie Algebras

The loop algebra $\mathbf{g}_t = \mathbf{g} \otimes \mathbb{C}[t, t^{-1}]$ is generated by the elements $j_a(n) = j_a \otimes t^n$ with defining commutator relations

$$[j_a(n), j_b(m)] = [j_a, j_b](n+m) \tag{8}$$

It may be centrally extended by adding to it a central algebra element k which may be treated as a constant since it commutes with all generators. Denoting the generators by capital letters we have the following commutator relations of the extended algebra

$$[J_a(n), J_b(m)] = f_{ab}{}^c J_c(n+m) + \kappa_{ab} k n \delta_{n+m,0} \tag{9}$$

By further including a derivation D satisfying

$$[D, J_a(n)] = n J_a(n) \qquad , \qquad [D, k] = 0 \tag{10}$$

one has obtained an affine Lie algebra of level $k^\vee$ where

$$k^\vee = \frac{2k}{\theta^2} \tag{11}$$

Note that whenever we consider affine Lie algebras in connection with the Virasoro algebra (through the Sugawara construction) we may take $D = -L_0$. Also note that the generators $\{J_a(n)\}$ are the modes of the currents $\bar{J}_a(z)$ in an affine current algebra

$$\begin{aligned} \bar{J}_a(z)\bar{J}_b(w) &= \frac{\kappa_{ab}k}{(z-w)^2} + \frac{f_{ab}{}^c\bar{J}_c(w)}{z-w} + \text{regular terms} \\ \bar{J}_a(z) &= \sum_{n\in\mathbb{Z}} J_a(n)z^{-n-1} \end{aligned} \tag{12}$$

The roots $\hat{\alpha}$ with respect to (H, k, D) are

$$\hat{\alpha} = \hat{\alpha}(n) = (\alpha, 0, n) \tag{13}$$

and

$$\hat{\alpha} = n\delta = (0, 0, n) \qquad , \qquad n \in \mathbb{Z} \setminus \{0\} \tag{14}$$

The simple roots are

$$\begin{aligned} \hat{\alpha}_i &= (\alpha_i, 0, 0) \qquad , \qquad 1 \leq i \leq r \\ \hat{\alpha}_0 &= (-\theta, 0, 1) \end{aligned} \tag{15}$$

and thus, the positive roots are

$$\hat{\alpha} = (\alpha, 0, n) \qquad , \qquad n > 0 \;\text{ or }\; (n = 0,\; \alpha > 0) \tag{16}$$

Weyl reflections (with respect to the roots (13)) of the weight lattice of an affine Lie algebra are defined by

$$\sigma_{\hat{\alpha}}(\Lambda) = \Lambda - \Lambda \cdot \hat{\alpha}^\vee \hat{\alpha} \tag{17}$$

where the scalar product of 2 elements $\Lambda = (\lambda, k, m)$ and $\Lambda' = (\lambda', k', m')$ is defined by

$$(\Lambda, \Lambda') = \Lambda \cdot \Lambda' = \lambda \cdot \lambda' + km' + k'm \tag{18}$$

Reflections with respect to the simple roots are sometimes written σ_i, $i = 0, ..., r$. It is obvious that $\sigma_{\hat{\alpha}}^{-1} = \sigma_{\hat{\alpha}}$, and it is easily seen that the Weyl reflection with respect to $\hat{\alpha} = (\alpha, 0, n)$ takes the weight $\Lambda = (\lambda, k, m)$ to

$$\begin{aligned} \sigma_{\hat{\alpha}}(\Lambda) &= \Big(\sigma_\alpha(\lambda_n(\alpha^\vee)), k, m + \frac{1}{2k}(\lambda^2 - \lambda_n^2(\alpha^\vee))\Big) \\ \lambda_n(\alpha^\vee) &= \lambda + nk\alpha^\vee \end{aligned} \tag{19}$$

Inspired by this one may introduce the translation operators $t_{\alpha^\vee}$ defined by

$$t_{\alpha^\vee}(\Lambda) = \Big(\lambda_1(\alpha^\vee), k, m + \frac{1}{2k}(\lambda^2 - \lambda_1^2(\alpha^\vee))\Big) \tag{20}$$

By induction it is seen that for all $n \in \mathbb{Z}$

$$t^n_{\alpha^\vee}(\Lambda) = \left(\lambda_n(\alpha^\vee), k, m + \frac{1}{2k}(\lambda^2 - \lambda_n^2(\alpha^\vee))\right) \tag{21}$$

The translation operators satisfy

$$t_{\alpha^\vee+\beta^\vee} = t_{\alpha^\vee} \circ t_{\beta^\vee} = t_{\beta^\vee} \circ t_{\alpha^\vee} \qquad , \qquad t_{-\alpha^\vee} = t^{-1}_{\alpha^\vee} \tag{22}$$

and in particular

$$\sigma_{\hat{\alpha}} = \sigma_\alpha \circ t^n_{\alpha^\vee} \tag{23}$$

where the definition of the Weyl reflection σ_α is trivially generalized to the following action on $r+2$ dimensional weights $\Lambda = (\lambda, k, m)$:

$$\sigma_\alpha(\Lambda) = \sigma_{(\alpha,0,0)}(\Lambda) = (\sigma_\alpha(\lambda), k, m) \tag{24}$$

Furthermore, the translation operator satisfies

$$t_{\sigma_\alpha(\beta^\vee)} = \sigma_\alpha \circ t_{\beta^\vee} \circ \sigma_\alpha \tag{25}$$

and from this it follows immediately that

$$t_{\alpha^\vee} = \sigma_{(-\alpha,0,1)} \circ \sigma_{(\alpha,0,0)} \tag{26}$$

and in particular

$$t_{\theta^\vee} = \sigma_0 \circ \sigma_\theta \tag{27}$$

Thus, in the case of $SL(2)$ where $r = 1$ and hence $\theta = \alpha_1$, a general (positive) affine Weyl reflection may be decomposed as

$$\sigma_{(\theta,0,n)} = \sigma_1 \circ (\sigma_0 \circ \sigma_1)^n \tag{28}$$

3 Decompositions of Affine Weyl Reflections

Let us first present the following lemma which is a key ingredient in our decomposition procedure of affine Weyl reflections

Lemma

$$t_{\alpha^\vee} = \sigma_0 \circ \sigma_{\theta-\alpha} \circ \sigma_0 \circ \sigma_\alpha \qquad , \qquad \theta^\vee \cdot \alpha = 1 \tag{29}$$

Even though the proof of the lemma is straightforward, it seems to be a new result. Note that the condition $\theta^\vee \cdot \alpha = 1$ implies $\theta - \alpha > 0$ (see e.g. [13]).

A general affine Weyl reflection $\sigma_{\hat{\alpha}}$, $\hat{\alpha} = (\alpha, 0, n)$, may be written

$$\sigma_{\hat{\alpha}} = \sigma_\alpha \circ t^n_{\alpha^\vee} = \sigma_\alpha \circ \left(t_{\alpha^\vee_{i_1}} \circ \ldots \circ t_{\alpha^\vee_{i_N}}\right)^n \tag{30}$$

where

$$\alpha^\vee = \alpha^\vee_{i_1} + \ldots + \alpha^\vee_{i_N} \tag{31}$$

is a unique (integer) expansion of the dual root $\alpha^\vee$ on dual simple roots $\alpha_i^\vee$. Here we have assumed that $\alpha > 0$. For $\hat{\alpha} = (-\alpha, 0, n)$ with $\alpha > 0$, we have

$$\sigma_{\hat{\alpha}} = \sigma_{-\alpha} \circ t^n_{-\alpha^\vee} = \sigma_\alpha \circ \left(t_{\alpha^\vee_{i_1}} \circ ... \circ t_{\alpha^\vee_{i_N}} \right)^{-n} \tag{32}$$

This reduces our first problem of decomposing the affine Weyl reflections in terms of fundamental reflections and classical Weyl reflections, to one of determining the translation operators with respect to dual simple roots.

Using the explicit representations of the root systems in Appendix A, it turns out that in the (generally non-unique) representation

$$\alpha_i^\vee = \theta^\vee - \beta_1^{(i)\vee} - ... - \beta_{M_i}^{(i)\vee} \tag{33}$$

of $\alpha_i^\vee$, one needs at most 2 dual positive roots $\beta_j^{(i)\vee}$, that is $M = 1, 2$ ($M = 0$ only in the simple case of $A_1 \simeq sl(2)$). This means that

$$t_{\alpha_i^\vee} = t_{\theta^\vee - \beta_1^{(i)\vee} - \beta_2^{(i)\vee}} = t^{-1}_{\beta_1^{(i)\vee}} \circ t^{-1}_{\beta_2^{(i)\vee}} \circ t_{\theta^\vee} \tag{34}$$

Example by example one may now attempt to use the lemma on $t_{\beta_1^\vee}$ (and for $M = 2$ also on $t_{\beta_2^\vee}$). It turns out that in most cases the lemma applies. However, for $t_{\alpha_i^\vee}$, $1 < i < r$, in C_r (and for $t_{\alpha_1^\vee}$ in C_r and for $t_{\alpha_2^\vee}$ in G_2, see below) it does not apply directly, but one may then repeat the procedure for $\beta_j^\vee = \theta^\vee - \gamma_1^\vee - ... - \gamma_{M'}^\vee$. Indeed we may write

$$\begin{aligned} \alpha_i^\vee &= \theta^\vee - (\alpha_{1i}^-)^\vee - \alpha_{i+1,i+1}^\vee \\ t_{\alpha_i^\vee} &= t^{-1}_{(\alpha_{1i}^-)^\vee} \circ t^{-1}_{\alpha^\vee_{i+1,i+1}} \circ t_{\theta^\vee} \end{aligned} \tag{35}$$

where $\theta^\vee \cdot \alpha_{1i}^- = 1$, but $\theta^\vee \cdot \alpha_{i+1,i+1} \neq 1$. In the second step we write

$$\begin{aligned} \alpha_{i+1,i+1}^\vee &= \theta^\vee - (\alpha_{1,i+1}^-)^\vee \qquad , \qquad \theta^\vee \cdot \alpha_{1,i+1}^- = 1 \\ t_{\alpha^\vee_{i+1,i+1}} &= t_{\theta^\vee} \circ t^{-1}_{(\alpha^-_{1,i+1})^\vee} \end{aligned} \tag{36}$$

and in conclusion we have

$$\begin{aligned} t_{\alpha_i^\vee} &= t^{-1}_{(\alpha_{1i}^-)^\vee} \circ t_{(\alpha^-_{1,i+1})^\vee} \\ &= \sigma_{\alpha_{1i}^-} \circ \sigma_0 \circ \sigma_{\alpha_{1i}^+} \circ \sigma_{\alpha^+_{1,i+1}} \circ \sigma_0 \circ \sigma_{\alpha^-_{1,i+1}} \end{aligned} \tag{37}$$

In the following Table 1 and Table 2 we have summarized our findings.

Table 1

$\mathbf{g}$	Fundamental translations	
A_r	$t_{\alpha_1^\vee} = \sigma_{\alpha_{2,r+1}} \circ \sigma_0 \circ \sigma_{\alpha_1} \circ \sigma_\theta$ $t_{\alpha_i^\vee} = \sigma_{\alpha_{1i}} \circ \sigma_0 \circ \sigma_{\alpha_{i,r+1}} \circ \sigma_0 \circ \sigma_{\alpha_{i+1,r+1}} \circ \sigma_0 \circ \sigma_{\alpha_{1,i+1}} \circ \sigma_\theta$ $t_{\alpha_r^\vee} = \sigma_{\alpha_{1r}} \circ \sigma_0 \circ \sigma_{\alpha_r} \circ \sigma_\theta$	$1 < i < r$
B_r	$t_{\alpha_1^\vee} = \sigma_{\alpha_{22}} \circ \sigma_0 \circ \sigma_{\alpha_{11}} \circ \sigma_\theta$ $t_{\alpha_2^\vee} = \sigma_{\alpha_{13}^+} \circ \sigma_0 \circ \sigma_{\alpha_2} \circ \sigma_\theta$ $t_{\alpha_i^\vee} = \sigma_{\alpha_{1i}^-} \circ \sigma_0 \circ \sigma_{\alpha_{2i}^+} \circ \sigma_0 \circ \sigma_{\alpha_{2,i+1}^+} \circ \sigma_0 \circ \sigma_{\alpha_{1,i+1}^-} \circ \sigma_\theta$ $t_{\alpha_r^\vee} = \sigma_{\alpha_{1r}^-} \circ \sigma_0 \circ \sigma_{\alpha_{2r}^+} \circ \sigma_0 \circ \sigma_{\alpha_{2r}^-} \circ \sigma_0 \circ \sigma_{\alpha_{1r}^+} \circ \sigma_\theta$	$2 < i < r$
C_r	$t_{\alpha_1^\vee} = \sigma_0 \circ \sigma_{\alpha_{12}^+} \circ \sigma_0 \circ \sigma_{\alpha_1}$ $t_{\alpha_i^\vee} = \sigma_{\alpha_{1i}^-} \circ \sigma_0 \circ \sigma_{\alpha_{1i}^+} \circ \sigma_{\alpha_{1,i+1}^+} \circ \sigma_0 \circ \sigma_{\alpha_{1,i+1}^-}$ $t_{\alpha_r^\vee} = \sigma_{\alpha_{1r}^-} \circ \sigma_0 \circ \sigma_{\alpha_{1r}^+} \circ \sigma_\theta$	$1 < i < r$
D_r	$t_{\alpha_1^\vee} = \sigma_{\alpha_{2r}^-} \circ \sigma_0 \circ \sigma_{\alpha_{1r}^+} \circ \sigma_0 \circ \sigma_{\alpha_{2r}^+} \circ \sigma_0 \circ \sigma_{\alpha_{1r}^-} \circ \sigma_\theta$ $t_{\alpha_2^\vee} = \sigma_{\alpha_{13}^+} \circ \sigma_0 \circ \sigma_{\alpha_2} \circ \sigma_\theta$ $t_{\alpha_i^\vee} = \sigma_{\alpha_{1i}^-} \circ \sigma_0 \circ \sigma_{\alpha_{2i}^+} \circ \sigma_0 \circ \sigma_{\alpha_{2,i+1}^+} \circ \sigma_0 \circ \sigma_{\alpha_{1,i+1}^-} \circ \sigma_\theta$ $t_{\alpha_r^\vee} = \sigma_{\alpha_{1,r-1}^-} \circ \sigma_0 \circ \sigma_{\alpha_{2,r-1}^+} \circ \sigma_0 \circ \sigma_{\alpha_{2r}^-} \circ \sigma_0 \circ \sigma_{\alpha_{1r}^+} \circ \sigma_\theta$	$2 < i < r$

Table 2

$\mathbf{g}$	Fundamental translations	
E_6	$t_{\alpha_1^\vee} = \sigma_{\alpha_{+--++}} \circ \sigma_0 \circ \sigma_{\alpha_{23}^+} \circ \sigma_0 \circ \sigma_{\alpha_{13}^+} \circ \sigma_0 \circ \sigma_{\alpha_{-+-++}} \circ \sigma_\theta$ $t_{\alpha_2^\vee} = \sigma_{\alpha_{++--+}} \circ \sigma_0 \circ \sigma_{\alpha_{34}^+} \circ \sigma_0 \circ \sigma_{\alpha_{24}^+} \circ \sigma_0 \circ \sigma_{\alpha_{+-+-+}} \circ \sigma_\theta$ $t_{\alpha_3^\vee} = \sigma_{\alpha_{+++--}} \circ \sigma_0 \circ \sigma_{\alpha_{45}^+} \circ \sigma_0 \circ \sigma_{\alpha_{35}^+} \circ \sigma_0 \circ \sigma_{\alpha_{++-+-}} \circ \sigma_\theta$ $t_{\alpha_4^\vee} = \sigma_{\alpha_{-+++-}} \circ \sigma_0 \circ \sigma_{\alpha_{15}^+} \circ \sigma_0 \circ \sigma_{\alpha_{14}^+} \circ \sigma_0 \circ \sigma_{\alpha_{-++-+}} \circ \sigma_\theta$ $t_{\alpha_5^\vee} = \sigma_{\alpha_{23}^+} \circ \sigma_0 \circ \sigma_{\alpha_{+--++}} \circ \sigma_0 \circ \sigma_{\alpha_{45}^+} \circ \sigma_0 \circ \sigma_{\alpha_{+++--}} \circ \sigma_\theta$ $t_{\alpha_6^\vee} = \sigma_{\alpha_{--+++}} \circ \sigma_0 \circ \sigma_{\alpha_6} \circ \sigma_\theta$	
E_7	$t_{\alpha_1^\vee} = \sigma_{\alpha_{+-++++}} \circ \sigma_0 \circ \sigma_{\alpha_{-+----}} \circ \sigma_0 \circ \sigma_{\alpha_{+-----}} \circ \sigma_0 \circ \sigma_{\alpha_{-+++++}} \circ \sigma_\theta$ $t_{\alpha_2^\vee} = \sigma_{\alpha_{++-+++}} \circ \sigma_0 \circ \sigma_{\alpha_{--+---}} \circ \sigma_0 \circ \sigma_{\alpha_{-+----}} \circ \sigma_0 \circ \sigma_{\alpha_{+-++++}} \circ \sigma_\theta$ $t_{\alpha_3^\vee} = \sigma_{\alpha_{+++-++}} \circ \sigma_0 \circ \sigma_{\alpha_{---+--}} \circ \sigma_0 \circ \sigma_{\alpha_{--+---}} \circ \sigma_0 \circ \sigma_{\alpha_{++-+++}} \circ \sigma_\theta$ $t_{\alpha_4^\vee} = \sigma_{\alpha_{++++-+}} \circ \sigma_0 \circ \sigma_{\alpha_{----+-}} \circ \sigma_0 \circ \sigma_{\alpha_{---+--}} \circ \sigma_0 \circ \sigma_{\alpha_{+++-++}} \circ \sigma_\theta$ $t_{\alpha_5^\vee} = \sigma_{\alpha_{+++++-}} \circ \sigma_0 \circ \sigma_{\alpha_{-----+}} \circ \sigma_0 \circ \sigma_{\alpha_{----+-}} \circ \sigma_0 \circ \sigma_{\alpha_{++++-+}} \circ \sigma_\theta$ $t_{\alpha_6^\vee} = \sigma_{\alpha_{-+++++}} \circ \sigma_0 \circ \sigma_{\alpha_6} \circ \sigma_\theta$ $t_{\alpha_7^\vee} = \sigma_{\alpha_{--+++-}} \circ \sigma_0 \circ \sigma_{\alpha_{++---+}} \circ \sigma_0 \circ \sigma_{\alpha_{-----+}} \circ \sigma_0 \circ \sigma_{\alpha_{+++++-}} \circ \sigma_\theta$	
E_8	$t_{\alpha_i^\vee} = \sigma_{\alpha_{i7}^+} \circ \sigma_0 \circ \sigma_{\alpha_{i8}^-} \circ \sigma_0 \circ \sigma_{\alpha_{i+1,8}^-} \circ \sigma_0 \circ \sigma_{\alpha_{i+1,7}^+} \circ \sigma_\theta$ $t_{\alpha_6^\vee} = \sigma_{\alpha_{68}^+} \circ \sigma_0 \circ \sigma_{\alpha_6} \circ \sigma_\theta$ $t_{\alpha_7^\vee} = \sigma_{\alpha_{--+++++}} \circ \sigma_0 \circ \sigma_{\alpha_{++----+}} \circ \sigma_0 \circ \sigma_{\alpha_{27}^+} \circ \sigma_0 \circ \sigma_{\alpha_{28}^-} \circ \sigma_\theta$ $t_{\alpha_8^\vee} = \sigma_{\alpha_{17}^-} \circ \sigma_0 \circ \sigma_{\alpha_{18}^+} \circ \sigma_0 \circ \sigma_{\alpha_{28}^-} \circ \sigma_0 \circ \sigma_{\alpha_{27}^+} \circ \sigma_\theta$	$1 \le i \le 5$
F_4	$t_{\alpha_1^\vee} = \sigma_{\alpha_{23}^+} \circ \sigma_0 \circ \sigma_{\alpha_{13}^-} \circ \sigma_0 \circ \sigma_{\alpha_{24}^+} \circ \sigma_0 \circ \sigma_{\alpha_{14}^-} \circ \sigma_\theta$ $t_{\alpha_2^\vee} = \sigma_{\alpha_{13}^+} \circ \sigma_0 \circ \sigma_{\alpha_2} \circ \sigma_\theta$ $t_{\alpha_3^\vee} = \sigma_{\alpha_{13}^-} \circ \sigma_0 \circ \sigma_{\alpha_{23}^+} \circ \sigma_0 \circ \sigma_{\alpha_{24}^+} \circ \sigma_0 \circ \sigma_{\alpha_{14}^-} \circ \sigma_\theta$ $t_{\alpha_4^\vee} = \sigma_{\alpha_{14}^-} \circ \sigma_0 \circ \sigma_{\alpha_{24}^+} \circ \sigma_0 \circ \sigma_{\alpha_{24}^-} \circ \sigma_0 \circ \sigma_{\alpha_{14}^+} \circ \sigma_\theta$	
G_2	$t_{\alpha_1^\vee} = \sigma_{\alpha_2} \circ \sigma_0 \circ \sigma_{\alpha_{+-+}} \circ \sigma_0 \circ \sigma_{\alpha_2} \circ \sigma_0 \circ \sigma_{\alpha_{+-+}} \circ \sigma_\theta$ $t_{\alpha_2^\vee} = \sigma_0 \circ \sigma_{\alpha_{+-+}} \circ \sigma_0 \circ \sigma_{\alpha_2}$	

Note that these expressions are neither unique, nor necessarily irreducible. First of all because our construction itself is not unique. Let us illustrate this by considering some examples. A simple one is provided by $t_{\alpha_1^\vee}$ in E_7 where $\alpha_1^\vee$ may be represented in several ways

$$\begin{aligned} \alpha_1^\vee &= \theta^\vee - \alpha^\vee_{+-abcd} - \alpha^\vee_{+-a'b'c'd'} \\ (a,b,c,d) &= -(a',b',c',d') \end{aligned} \tag{38}$$

Here there is an even number of minus signs amongst the signs a, b, c, d, and therefore also amongst the signs a', b', c', d'. The result in Table 2 is based on the choice $a = b = c = d = +$. More generally, the commutativity of the translation operators immediately spoils an a priori possible uniqueness of the procedure. Furthermore, a very simple alternative to the general procedure for obtaining $t_{\alpha_i^\vee}$ exists when $\alpha_i \cdot \theta^\vee = 1$, since in that case the lemma applies directly on α_i. In the following Table 3 we have listed those simple roots.

Table 3

$\mathbf{g}$	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$\alpha_i \;:\; \alpha_i \cdot \theta^\vee = 1$	$\alpha_1,\ \alpha_r$	α_2	α_1	α_2	α_6	α_6	α_6	α_2	α_2

It will depend on the application which representation of the corresponding translation operator $t_{\alpha_i^\vee}$ that is superior. Let us finally comment on G_2 where the procedure does not apply directly due to

$$t_{\theta^\vee} = t_{\alpha_1^\vee} \circ t^2_{\alpha_2^\vee} \tag{39}$$

However, $\alpha_2 \cdot \theta^\vee = 1$ so we may simply use the lemma, the result of which is in Table 2 of fundamental translations above. Similarly, the result for $t_{\alpha_1^\vee}$ in C_r is obtained directly from the lemma, c.f. Table 3 above.

3.1 Decompositions of Classical Weyl Reflections

It remains to account for how to decompose the classical Weyl reflections. For this purpose we use that if $\beta \cdot \alpha^\vee > 0$ then $\beta \cdot \alpha^\vee = 1, 2, 3$ and $\beta - \alpha \in \Delta$, and that from (7) it follows that

$$\sigma_\beta = \sigma_\alpha \circ \sigma_{\beta - \beta\cdot\alpha^\vee\alpha} \circ \sigma_\alpha \tag{40}$$

Let us illustrate our procedure for decomposing an arbitrary Weyl reflection by considering the case of B_r where the root type α^-_{ij} may be expanded as

$$\alpha^-_{ij} = \alpha_{j-1} + ... + \alpha_i \tag{41}$$

We see that

$$(\alpha_{j-1} + ... + \alpha_{i'}) \cdot \alpha^\vee_{i'} = 1 \qquad , \qquad i' = i, ..., j-2 \tag{42}$$

so according to (40) the decomposition in Table 4 immediately follows. Similarly, the expansion

$$\alpha_{ii} = \alpha_r + ... + \alpha_i \tag{43}$$

leads to the decomposition of $\sigma_{\alpha_{ii}}$. Finally, we have

$$\alpha_{ij}^+ = \alpha_{ii} + \alpha_{jj} \qquad , \qquad \alpha_{ij}^+ \cdot \alpha_{jj}^\vee = 2 \tag{44}$$

so according to (40)

$$\sigma_{\alpha_{ij}^+} = \sigma_{\alpha_{jj}} \circ \sigma_{\alpha_{ij}^+ - 2\alpha_{jj}} \circ \sigma_{\alpha_{jj}} = \sigma_{\alpha_{jj}} \circ \sigma_{\alpha_{ij}^-} \circ \sigma_{\alpha_{jj}} \tag{45}$$

In Table 4, Table 5 and Table 6 we have summarized our findings for all simple finite dimensional Lie algebras. Note that for notational reasons, the decomposition of a Weyl reflection σ_α may depend on other decompositions given (above it) in the tables.

Table 4

g	Decompositions of classical Weyl reflections	
A_r	$\sigma_{\alpha_{ij}} = \sigma_i \circ ... \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ ... \circ \sigma_i$	
B_r	$\sigma_{\alpha_{ij}^-} = \sigma_i \circ ... \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ ... \circ \sigma_i$	
	$\sigma_{\alpha_{ii}} = \sigma_i \circ ... \circ \sigma_{r-1} \circ \sigma_r \circ \sigma_{r-1} \circ ... \circ \sigma_i$	$i \neq r$
	$\sigma_{\alpha_{ij}^+} = \sigma_{\alpha_{jj}} \circ \sigma_{\alpha_{ij}^-} \circ \sigma_{\alpha_{jj}}$	
C_r	$\sigma_{\alpha_{ij}^-} = \sigma_i \circ ... \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ ... \circ \sigma_i$	
	$\sigma_{\alpha_{ir}^+} = \sigma_r \circ ... \circ \sigma_i \circ ... \circ \sigma_r$	
	$\sigma_{\alpha_{ij}^+} = \sigma_j \circ ... \circ \sigma_{r-1} \circ \sigma_{\alpha_{ir}^+} \circ \sigma_{r-1} \circ ... \circ \sigma_j$	$j < r$
	$\sigma_{\alpha_{ii}} = \sigma_{\alpha_{ir}^-} \circ \sigma_r \circ \sigma_{\alpha_{ir}^-}$	$i \neq r$
D_r	$\sigma_{\alpha_{ij}^-} = \sigma_i \circ ... \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ ... \circ \sigma_i$	
	$\sigma_{\alpha_{ir}^+} = \sigma_r \circ \sigma_{r-2} \circ ... \circ \sigma_i \circ ... \circ \sigma_{r-2} \circ \sigma_r$	$i < r-1$
	$\sigma_{\alpha_{i,r-1}^+} = \sigma_r \circ \sigma_{r-1} \circ ... \circ \sigma_i \circ ... \circ \sigma_{r-1} \circ \sigma_r$	
	$\sigma_{\alpha_{ij}^+} = \sigma_j \circ ... \circ \sigma_{r-2} \circ \sigma_{\alpha_{i,r-1}^+} \circ \sigma_{r-2} \circ ... \circ \sigma_j$	$j < r-1$

Table 5

g	Decompositions of classical Weyl reflections	
E_6	$\sigma_{\alpha^-_{ij}} = \sigma_i \circ ... \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ ... \circ \sigma_i$	
	$\sigma_{\alpha^+_{1j}} = \sigma_6 \circ \sigma_{\alpha^-_{2j}} \circ \sigma_6$	$3 \leq j \leq 5$
	$\sigma_{\alpha^+_{2j}} = \sigma_6 \circ \sigma_{\alpha^-_{1j}} \circ \sigma_6$	
	$\sigma_{\alpha^+_{3j}} = \sigma_2 \circ \sigma_{\alpha^+_{2j}} \circ \sigma_2$	
	$\sigma_{\alpha^+_{45}} = \sigma_3 \circ \sigma_{\alpha^+_{35}} \circ \sigma_3$	
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm}} \big\|_{(\text{one } +, \text{ position } i)} = \sigma_{\alpha^-_{1i}} \circ \sigma_5 \circ \sigma_{\alpha^-_{1i}}$	$i \neq 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm}} \big\|_{(\text{three } +, \text{ positions } i<j<k)} = \sigma_{\alpha^+_{jk}} \circ \sigma_5 \circ \sigma_{\alpha^+_{jk}}$	$i = 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm}} \big\|_{(\text{three } +, \text{ positions } i<j<k)} = \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^-_{1i}} \circ \sigma_5 \circ \sigma_{\alpha^-_{1i}} \circ \sigma_{\alpha^+_{jk}}$	$i \neq 1$
	$\sigma_{\alpha_{+++++}} = \sigma_{\alpha^+_{45}} \circ \sigma_{\alpha^+_{23}} \circ \sigma_5 \circ \sigma_{\alpha^+_{23}} \circ \sigma_{\alpha^+_{45}}$	
E_7	$\sigma_{\alpha^-_{ij}} = \sigma_i \circ ... \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ ... \circ \sigma_i$	
	$\sigma_{\alpha^+_{1j}} = \sigma_7 \circ \sigma_{\alpha^-_{2j}} \circ \sigma_7$	$3 \leq j \leq 6$
	$\sigma_{\alpha^+_{2j}} = \sigma_7 \circ \sigma_{\alpha^-_{1j}} \circ \sigma_7$	
	$\sigma_{\alpha^+_{3j}} = \sigma_2 \circ \sigma_{\alpha^+_{2j}} \circ \sigma_2$	
	$\sigma_{\alpha^+_{4j}} = \sigma_3 \circ \sigma_{\alpha^+_{3j}} \circ \sigma_3$	
	$\sigma_{\alpha^+_{56}} = \sigma_4 \circ \sigma_{\alpha^+_{46}} \circ \sigma_4$	
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm}} \big\|_{(\text{one } +, \text{ position } i)} = \sigma_{\alpha^-_{1i}} \circ \sigma_6 \circ \sigma_{\alpha^-_{1i}}$	$i \neq 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm}} \big\|_{(\text{three } +, \text{ positions } i<j<k)} = \sigma_{\alpha^+_{jk}} \circ \sigma_6 \circ \sigma_{\alpha^+_{jk}}$	$i = 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm}} \big\|_{(\text{three } +, \text{ positions } i<j<k)} = \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^-_{1i}} \circ \sigma_6 \circ \sigma_{\alpha^-_{1i}} \circ \sigma_{\alpha^+_{jk}}$	$i \neq 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm}} \big\|_{(\text{five } +, \text{ positions } i<j<k<l<m)} = \sigma_{\alpha^+_{lm}} \circ \sigma_{\alpha^+_{jk}} \circ \sigma_6 \circ \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^+_{lm}}$	$i = 1$
	$\sigma_{\alpha_{-+++++}} = \sigma_{\alpha^+_{56}} \circ \sigma_{\alpha^+_{34}} \circ \sigma_{\alpha^-_{12}} \circ \sigma_6 \circ \sigma_{\alpha^-_{12}} \circ \sigma_{\alpha^+_{34}} \circ \sigma_{\alpha^+_{56}}$	
	$\sigma_\theta = \sigma_6 \circ \sigma_{\alpha_{-+++++}} \circ \sigma_6$	

Table 6

g	Decompositions of classical Weyl reflections	
E_8	$\sigma_{\alpha^-_{ij}} = \sigma_i \circ \ldots \circ \sigma_{j-2} \circ \sigma_{j-1} \circ \sigma_{j-2} \circ \ldots \circ \sigma_i$	$j \neq 8$
	$\sigma_{\alpha^+_{1j}} = \sigma_8 \circ \sigma_{\alpha^-_{2j}} \circ \sigma_8$	$3 \leq j \leq 7$
	$\sigma_{\alpha^+_{2j}} = \sigma_8 \circ \sigma_{\alpha^-_{1j}} \circ \sigma_8$	$3 \leq j \leq 7$
	$\sigma_{\alpha^+_{ij}} = \sigma_{i-1} \circ \sigma_{\alpha^+_{i-1,j}} \circ \sigma_{i-1}$	$3 \leq i \leq 6$ $j \neq 8$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm\pm\pm}} \big\vert_{(\text{one } +,\text{ position } i)} = \sigma_{\alpha^-_{1i}} \circ \sigma_7 \circ \sigma_{\alpha^-_{1i}}$	$i \neq 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm\pm\pm}} \big\vert_{(\text{three } +,\text{ positions } i<j<k)} = \sigma_{\alpha^+_{jk}} \circ \sigma_7 \circ \sigma_{\alpha^+_{jk}}$	$i = 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm\pm\pm}} \big\vert_{(\text{three } +,\text{ positions } i<j<k)} = \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^-_{1i}} \circ \sigma_7 \circ \sigma_{\alpha^-_{1i}} \circ \sigma_{\alpha^+_{jk}}$	$i \neq 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm\pm\pm}} \big\vert_{(\text{five } +,\text{ positions } i<j<k<l<m)} = \sigma_{\alpha^+_{lm}} \circ \sigma_{\alpha^+_{jk}} \circ \sigma_7 \circ \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^+_{lm}}$	$i = 1$
	$\sigma_{\alpha_{\pm\pm\pm\pm\pm\pm\pm\pm}} \big\vert_{(\text{five } +,\text{ positions } i<j<k<l<m)}$ $= \sigma_{\alpha^+_{lm}} \circ \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^-_{1i}} \circ \sigma_7 \circ \sigma_{\alpha^-_{1i}} \circ \sigma_{\alpha^+_{jk}} \circ \sigma_{\alpha^+_{lm}}$	$i \neq 1$
	$\sigma_{\alpha_{++++++++}} = \sigma_{\alpha^+_{67}} \circ \sigma_{\alpha^+_{45}} \circ \sigma_{\alpha^+_{23}} \circ \sigma_7 \circ \sigma_{\alpha^+_{23}} \circ \sigma_{\alpha^+_{45}} \circ \sigma_{\alpha^+_{67}}$	
	$\sigma_{\alpha^-_{78}} = \sigma_7 \circ \sigma_{\alpha_{-++++++-}} \circ \sigma_7$	
	$\sigma_{\alpha^-_{i8}} = \sigma_i \circ \sigma_{\alpha^-_{i+1,8}} \circ \sigma_i$	$i < 7$
	$\sigma_{\alpha^+_{18}} = \sigma_7 \circ \sigma_{\alpha_{++++++++}} \circ \sigma_7$	
	$\sigma_{\alpha^+_{i8}} = \sigma_{i-1} \circ \sigma_{\alpha^+_{i-1,8}} \circ \sigma_{i-1}$	$2 \leq i \leq 7$
F_4	$\sigma_{\alpha_{--+}} = \sigma_4 \circ \sigma_1 \circ \sigma_4$	$\sigma_{\alpha_{-+-}} = \sigma_3 \circ \sigma_{\alpha_{--+}} \circ \sigma_3$
	$\sigma_{\alpha_{-++}} = \sigma_4 \circ \sigma_{\alpha_{-+-}} \circ \sigma_4$	$\sigma_{\alpha_{+-+}} = \sigma_2 \circ \sigma_{\alpha_{-++}} \circ \sigma_2$
	$\sigma_{\alpha_{++-}} = \sigma_3 \circ \sigma_{\alpha_{+-+}} \circ \sigma_3$	$\sigma_{\alpha_{+++}} = \sigma_4 \circ \sigma_{\alpha_{++-}} \circ \sigma_4$
	$\sigma_{\alpha_{+--}} = \sigma_2 \circ \sigma_{\alpha_{-+-}} \circ \sigma_2$	
	$\sigma_{\alpha_{11}} = \sigma_1 \circ \sigma_{\alpha_{+++}} \circ \sigma_1$	$\sigma_{\alpha_{22}} = \sigma_{\alpha_{-++}} \circ \sigma_{\alpha_{+++}} \circ \sigma_{\alpha_{-++}}$
	$\sigma_{\alpha_{33}} = \sigma_{\alpha_{+-+}} \circ \sigma_{\alpha_{+++}} \circ \sigma_{\alpha_{+-+}}$	
	$\sigma_{\alpha^-_{24}} = \sigma_3 \circ \sigma_2 \circ \sigma_3$	$\sigma_{\alpha^-_{12}} = \sigma_{\alpha_{--+}} \circ \sigma_3 \circ \sigma_{\alpha_{--+}}$
	$\sigma_{\alpha^-_{13}} = \sigma_{\alpha_{--+}} \circ \sigma_{\alpha^-_{24}} \circ \sigma_{\alpha_{--+}}$	$\sigma_{\alpha^-_{14}} = \sigma_{\alpha_{-+-}} \circ \sigma_2 \circ \sigma_{\alpha_{-+-}}$
	$\sigma_{\alpha^+_{ij}} = \sigma_{\alpha_{jj}} \circ \sigma_{\alpha^-_{ij}} \circ \sigma_{\alpha_{jj}}$	
G_2	$\sigma_{\alpha_{13}} = \sigma_2 \circ \sigma_1 \circ \sigma_2$	$\sigma_{\alpha_{23}} = \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1$
	$\sigma_{\alpha_{+-+}} = \sigma_1 \circ \sigma_2 \circ \sigma_1$	$\sigma_\theta = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2$

Obviously, these decompositions are not unique. In Table 4, Table 5 and Table 6 we have only listed one solution for each type of roots and not included all the fundamental Weyl reflections themselves.

4 Conclusions

In this paper we have provided explicit decompositions of affine Weyl reflections in terms of fundamental reflections. Together with the differential operator realizations of the Lie algebras [11, 12], this result allows generalizing the work [7] on fusion rules in affine $SL(2)$ current algebra to more general affine current algebras. We intend to come back elsewhere with a discussion along these lines [6].

Generalizations to superalgebras are also interesting. Recently, we have presented differential operator realizations of basic Lie superalgebras [14]. However, in order to determine the fusion rules in affine current superalgebras along the lines of Awata and Yamada [7] we need to work out not only decompositions of affine Weyl (super-)reflections but also Malikov-Feigin-Fuks type [2] expressions for the singular vectors. Studies of these issues would be interesting to undertake in future works.

Acknowledgement

The author thanks V.K. Dobrev for a copy of the unpublished preprint [4] and for pointing out reference [5].

A Embedding of Root Systems

In this appendix we review the well known embeddings (see e.g. [3]) of the root systems in $\mathbb{R}^n$ by means of an orthonormal basis $\{e_i\}_{i=1,\ldots,n}$. In the case of $\mathbf{g} = B_r, C_r, D_r, E_8, F_4$ we have $n = r$, in the case of $\mathbf{g} = A_r, E_7, G_2$ we have $n = r+1$ while for $\mathbf{g} = E_6$ $n = r+2$. The normalizations of the root systems are such that the short roots have length squared equal to 2, except for B_r and F_4 where the long roots have length squared equal to 2. However, a fixing of the root lengths here is of no importance for the general arguments in the main body of this paper where we keep the normalization of the root system (θ^2) a free parameter. The explicit embeddings are given in the following tables, Table 7, Table 8 and Table 9.

Table 7

$\mathbf{g}$	Simple roots	Δ_+
A_r	$\alpha_i = e_i - e_{i+1}$ $\quad 1 \leq i \leq r$	$\alpha_{ij} = e_i - e_j$ $\quad 1 \leq i < j \leq r+1$
		$\theta = \sum_{i=1}^{r} \alpha_i = e_1 - e_{r+1}$
B_r	$\alpha_i = e_i - e_{i+1}$ $\quad 1 \leq i < r$ $\alpha_r = e_r$	$\alpha_{ij}^{\pm} = e_i \pm e_j$ $\quad 1 \leq i < j \leq r$ $\alpha_{ii} = e_i$ $\quad 1 \leq i \leq r$
		$\theta = \alpha_1 + 2\sum_{i=2}^{r} \alpha_i = e_1 + e_2$
C_r	$\alpha_i = e_i - e_{i+1}$ $\quad 1 \leq i < r$ $\alpha_r = 2e_r$	$\alpha_{ij}^{\pm} = e_i \pm e_j$ $\quad 1 \leq i < j \leq r$ $\alpha_{ii} = 2e_i$ $\quad 1 \leq i \leq r$
		$\theta = 2\sum_{i=1}^{r-1} \alpha_i + \alpha_r = 2e_1$
D_r	$\alpha_i = e_i - e_{i+1}$ $\quad 1 \leq i < r$ $\alpha_r = e_{r-1} + e_r$	$\alpha_{ij}^{\pm} = e_i \pm e_j$ $\quad 1 \leq i < j \leq r$
		$\theta = \alpha_1 + 2\sum_{i=2}^{r-2} \alpha_i + \alpha_{r-1} + \alpha_r$ $= e_1 + e_2$

Table 8

$\mathbf{g}$	Simple roots	Δ_+
E_6	$\alpha_i = -e_i + e_{i+1} \qquad 1 \le i \le 4$ $\alpha_5 = \frac{1}{2}\left(e_1 - \sum_{j=2}^{7} e_j + e_8\right)$ $\alpha_6 = e_1 + e_2$	$\alpha_{ij}^{\pm} = \pm e_i + e_j \qquad 1 \le i < j \le 5$ $\alpha_{\pm\pm\pm\pm\pm} = \frac{1}{2}\left(\sum_{j=1}^{5}(\pm)e_j - e_6 - e_7 + e_8\right)$ even number of minus signs
		$\theta = 2\alpha_1 + 3\alpha_2 + 2\alpha_3$ $+ \alpha_4 + \alpha_5 + 2\alpha_6$ $= \frac{1}{2}\left(\sum_{j=1}^{5} e_j - e_6 - e_7 + e_8\right)$
E_7	$\alpha_i = -e_i + e_{i+1} \qquad 1 \le i \le 5$ $\alpha_6 = \frac{1}{2}\left(e_1 - \sum_{j=2}^{7} e_j + e_8\right)$ $\alpha_7 = e_1 + e_2$	$\alpha_{ij}^{\pm} = \pm e_i + e_j \qquad 1 \le i < j \le 6$ $\alpha_{78} = -e_7 + e_8$ $\alpha_{\pm\pm\pm\pm\pm\pm} = \frac{1}{2}\left(\sum_{j=1}^{6}(\pm)e_j - e_7 + e_8\right)$ odd number of minus signs
		$\theta = 3\alpha_1 + 4\alpha_2 + 3\alpha_3$ $+ 2\alpha_4 + \alpha_5 + 2\alpha_6 + 2\alpha_7$ $= -e_7 + e_8$
E_8	$\alpha_i = -e_i + e_{i+1} \qquad 1 \le i \le 6$ $\alpha_7 = \frac{1}{2}\left(e_1 - \sum_{j=2}^{7} e_j + e_8\right)$ $\alpha_8 = e_1 + e_2$	$\alpha_{ij}^{\pm} = \pm e_i + e_j \qquad 1 \le i < j \le 8$ $\alpha_{\pm\pm\pm\pm\pm\pm\pm} = \frac{1}{2}\left(\sum_{j=1}^{7}(\pm)e_j + e_8\right)$ even number of minus signs
		$\theta = 4\alpha_1 + 6\alpha_2 + 5\alpha_3 + 4\alpha_4$ $+ 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + 3\alpha_8$ $= e_7 + e_8$

Table 9

$\mathbf{g}$	Simple roots	Δ_+
F_4	$\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ $\alpha_2 = e_2 - e_3$ $\alpha_3 = e_3 - e_4$ $\alpha_4 = e_4$	$\alpha_{\pm\pm\pm} = \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$ $\alpha_{ij}^{\pm} = e_i \pm e_j \qquad 1 \leq i < j \leq 4$ $\alpha_{ii} = e_i \qquad 1 \leq i \leq 4$
		$\theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4$ $= e_1 + e_2$
G_2	$\alpha_1 = e_1 - e_2$ $\alpha_2 = -2e_1 + e_2 + e_3$	$\alpha_1,\ \alpha_2$ $\alpha_{13} = -e_1 + e_3,\ \alpha_{23} = -e_2 + e_3$ $\alpha_{+-+} = e_1 - 2e_2 + e_3$ $\alpha_{--+} = -e_1 - e_2 + 2e_3$
		$\theta = 3\alpha_1 + 2\alpha_2 = -e_1 - e_2 + 2e_3$

References

[1] V.G. Kac, *Infinite Dimensional Lie Algebras*, (3rd edition, Cambridge Univ. Press, 1990)

[2] F.G. Malikov, B.L. Feigin and D.B. Fuks, Funkt. Anal. Prilozhen **20** (1986) 25

[3] J. Fuchs, *Affine Lie Algebras and Quantum Groups*, (Cambridge Univ. Press, 1992)

[4] V.K. Dobrev, *Multiplet Classification of the Indecomposable Highest Weight Modules over Affine Lie Algebras and Invariant Differential Operators: the $A_l^{(1)}$ Example*, IC/85/9, preprint

[5] V.K. Dobrev and E. Sezgin, Int. Jour. Mod. Phys. **A 6** (1991) 4699

[6] J. Rasmussen, *Fusion Rules in Affine Current Algebras*, work in progress

[7] H. Awata and Y. Yamada, Mod. Phys. Lett. **A7** (1992) 1185

[8] V.G. Kac and D.A. Kazhdan, Adv. Math. **34** (1979) 97

[9] V.G. Kac and M. Wakimoto, Proc. Natl. Acad. Sci. USA **85** (1988) 4956;
V.G. Kac, and D.A. Kazhdan, Adv. Ser. Math. Phys., Vol. 7 (World Scientific, 1989), p. 138

[10] I.P. Ennes and A.V. Ramallo, *Fusion Rules and Singular Vectors of the osp*$(1|2)$ *Current Algebra*, US-FT-12/97, hep-th/9704065, preprint

[11] J. Rasmussen, *Applications of Free Fields in 2D Current Algebra*, Ph.D. thesis (The Niels Bohr Institute, 1996), hep-th/9610167

[12] J.L. Petersen, J. Rasmussen and M. Yu, *Free Field Realizations of 2D Current Algebras, Screening Currents and Primary Fields*, NBI-HE-97-12, AS-ITP-97-10, hep-th/9704052, accepted for publication in Nucl. Phys. B

[13] R.V. Moody and A. Pianzola, *Lie Algebras with Triangular Decompositions*, (John Wiley and Sons, 1995)

[14] J. Rasmussen, *Free Field Realizations of Affine Current Superalgebras, Screening Currents and Primary Fields*, NBI-HE-97-15, hep-th/9706091, preprint